Finding tiling spaces in the most curious places.

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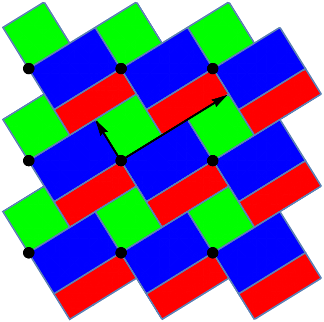
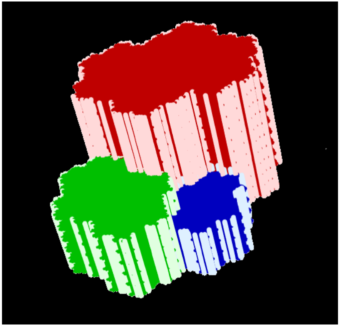
A partition of a 2-torus

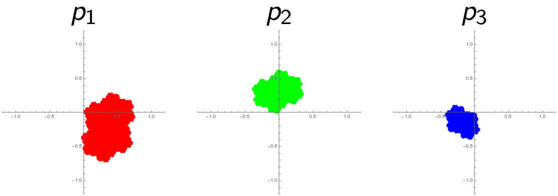


A partition of a 3-torus



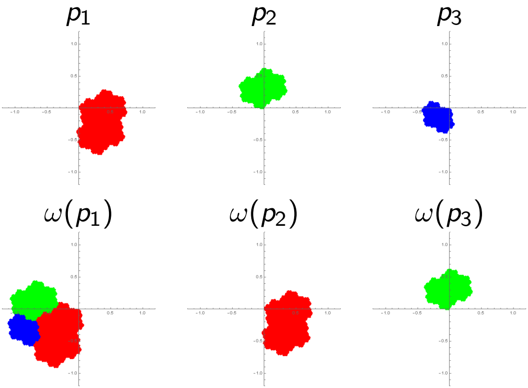
Substitution tiling systems, $(\Omega, \mathcal{P}, \omega)$

Prototiles, $\mathcal{P} = \{p_1, p_2 \ldots, p_n\}$. Each $p_i \subseteq \mathbb{R}^d$ is the closure of its interior.

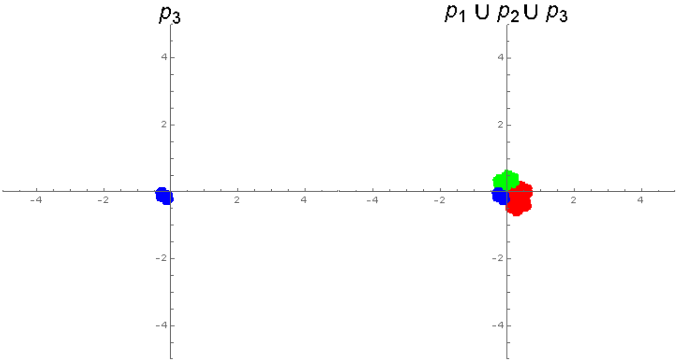


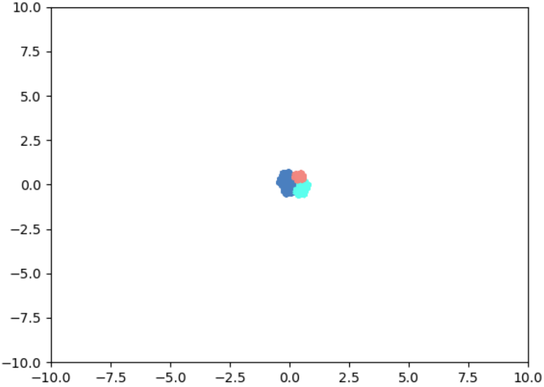
A tile t is a translation of some prototile.

A substitution rule $\omega(p_i)$ that inflates, possibly rotates and subdivides with translates of prototiles.

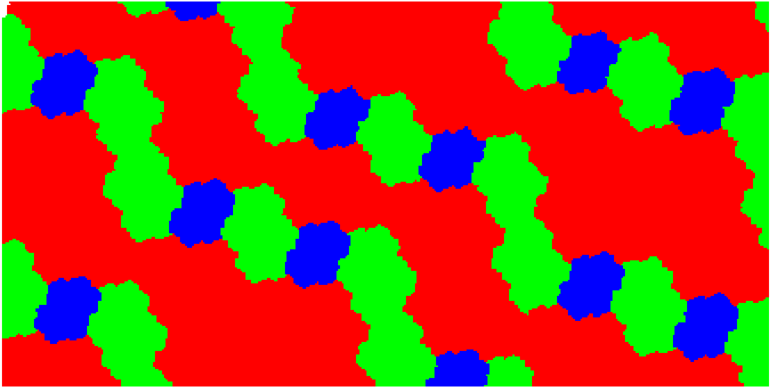


A partial tiling is a collection of tiles whose interiors are pairwise disjoint. A tiling is a partial tiling whose union is $\mathbb{R}^d$.
The substitution can be iterated and extended to all tilings.




10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

We define Ω to be the set of tilings T such that if $P \subseteq T$ then $P \subseteq \omega^k(t)$ for some tile t.

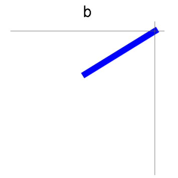
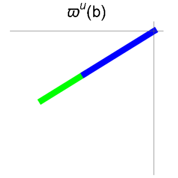
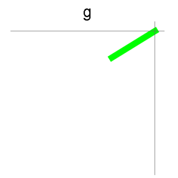
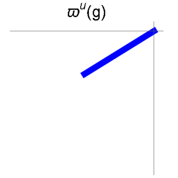


Forcing the border

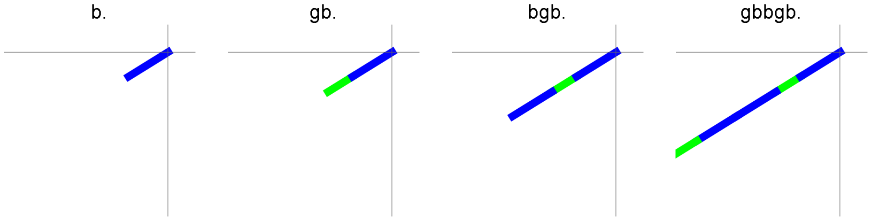
Definition

A tiling system $(\Omega, \mathcal{P}, \omega)$ forces its border if there is a $k \geq 1$ such that, if T and T' are two tilings containing a tile t, then the patches in $\omega^k(T)$ and $\omega^k(T')$ consisting of all tiles which meet $\omega^k(t)$ are identical.

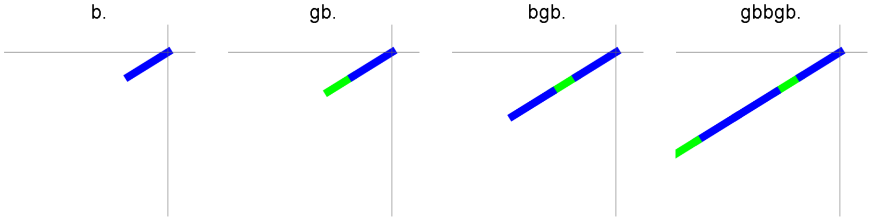
Fibonacci

$\omega^u : b. \mapsto gb.$		
$\omega^u : g. \mapsto b.$		

ω^u: Does not force border ($a \mapsto ba$ and $b \mapsto a$)



ω^u: Does not force border ($a \mapsto ba$ and $b \mapsto a$)



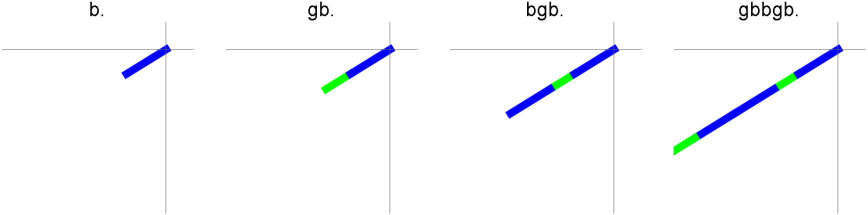
Extend on the right by b

$\dots gbgbb\boxed{\text{g}}b.\boxed{\text{b}}\dots$

Extend on the right by g

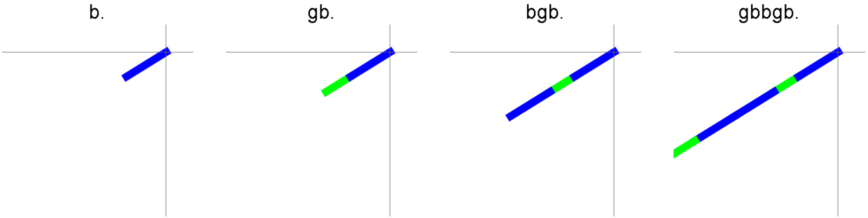
$\dots babaa\boxed{\text{g}}b.\boxed{\text{g}}\dots$

ω^u: Does not force border ($a \mapsto ba$ and $b \mapsto a$)



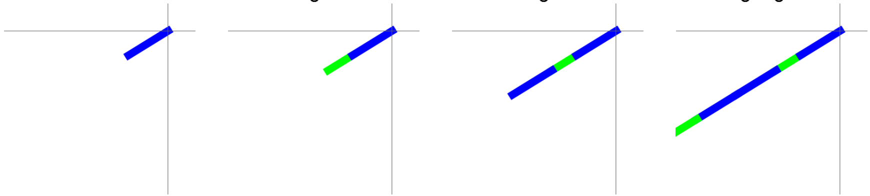
	Extend on the right by b	Extend on the right by g
	$\ldots gbgbb\boxed{\mathrm{g}}b.\boxed{\mathrm{b}}\ \ldots$	$\ldots babaa\boxed{\mathrm{g}}b.\boxed{\mathrm{g}}\ldots$
ω^u	$\ldots gbgb\boxed{\mathrm{b}}gb.\boxed{\mathrm{g}}b\ldots$	$\ldots bgbg\boxed{\mathrm{b}}gb.\boxed{\mathrm{b}}\ldots$

ω^u: Does not force border ($a \mapsto ba$ and $b \mapsto a$)



	Extend on the right by b	Extend on the right by g
	$\ldots gbgbb\boxed{\text{g}}b.\boxed{\text{b}}\ \ldots$	$\ldots babaa\boxed{\text{g}}b.\boxed{\text{g}}\ldots$
ω^u	$\ldots gbgb\boxed{\text{b}}gb.\boxed{\text{g}}b\ldots$	$\ldots bgbg\boxed{\text{b}}gb.\boxed{\text{b}}\ldots$
$(\omega^u)^2$	$\ldots gbg\boxed{\text{b}}bgb.\boxed{\text{b}}gb\ldots$	$\ldots gbg\boxed{\text{b}}bgb.\boxed{\text{g}}b\ldots$

ω^u: Does not force border ($a \mapsto ba$ and $b \mapsto a$)



	Extend on the right by b	Extend on the right by g
	$\ldots gbgbb\boxed{\text{g}}b.\boxed{\text{b}}\ \ldots$	$\ldots babaa\boxed{\text{g}}b.\boxed{\text{g}}\ldots$
ω^u	$\ldots gbgb\boxed{\text{b}}gb.\boxed{\text{g}}b\ldots$	$\ldots bgbg\boxed{\text{b}}gb.\boxed{\text{b}}\ \ldots$
$(\omega^u)^2$	$\ldots gbg\boxed{\text{b}}bgb.\boxed{\text{b}}gb\ldots$	$\ldots gbg\boxed{\text{b}}bgb.\boxed{\text{g}}b\ldots$
$(\omega^u)^3$	$\ldots g\boxed{\text{b}}gbbgb.\boxed{\text{g}}bgb\ldots$	$\ldots g\boxed{\text{b}}gbbgb.\boxed{\text{b}}gb\ldots$

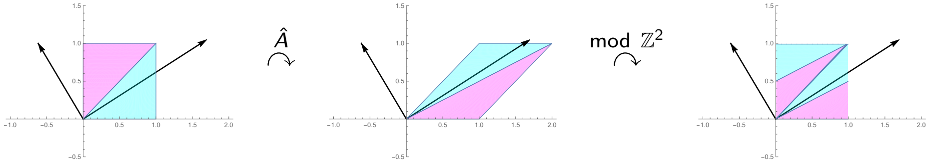
In 1995 Anderson and Putnam showed that if a aperiodic substitution tiling space forces its border then it is topologically conjugate to a solenoid.

Hyperbolic toral automorphism

Take $\hat{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Define $A : \mathbb{T}^2 \to \mathbb{T}^2$ as $A([x]) = [\hat{A}x]$. where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

A is a toral automorphism.



Eigenvalues : γ and $-\gamma^{-1}$, where $\gamma = \frac{1+\sqrt{5}}{2} > 1$.

Eigenvectors: $v_u = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$ and $v_s = \begin{bmatrix} -\gamma^{-1} \\ 1 \end{bmatrix}$.

$$\text{Notice } \mathbb{R}^2 = E^u \oplus E^s$$

Eigenvalues : γ and $-\gamma^{-1}$, where $\gamma = \frac{1+\sqrt{5}}{2} > 1$.

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$$\text{Notice} \quad \mathbb{R}^2 = E^u \oplus E^s$$

For general $\hat{A}$ in $GL_d(\mathbb{R})$ we define,

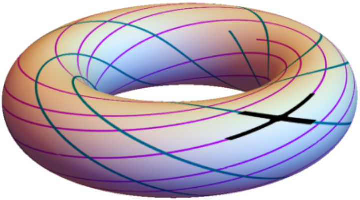
$$E^s = \{x \in \mathbb{R}^d \mid \|\hat{A}^n x\| \to 0, \ n \to +\infty\}$$
$$E^u = \{x \in \mathbb{R}^d \mid \|\hat{A}^n x\| \to 0, \ n \to -\infty\}$$

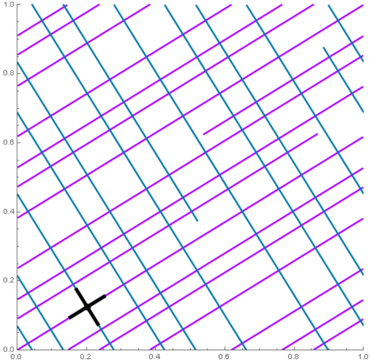
Definition

We say a matrix $\hat{A}$ is hyperbolic if $\hat{A}$ is in $GL_d(\mathbb{R})$ and,

$$\mathbb{R}^d = E^s \oplus E^u.$$

On a hyperbolic toral automorphism the global unstable and stable sets wrap around densely.





The local stable and unstable sets are given by moving a little bit along the eigendirections. Locally, $\mathbb{T}^2$ can be viewed as $\mathbb{R} \times \mathbb{R}$.

Let f be a homeomorphism.

Definition

We say two points x, y in X are stably equivalent and write $x \stackrel{s}{\sim} y$ if

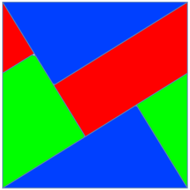
$$\lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0$$

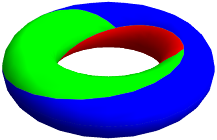
We let $X^s(x)$, the set of y with $x \stackrel{s}{\sim} y$.

Unstably equivalent points are defined analogously.

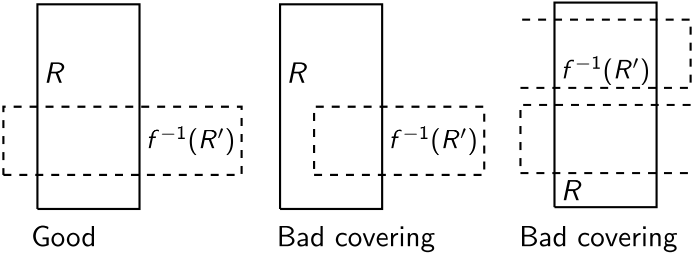
The HTA can be modeled using symbolic dynamics by way of Markov partitions by a finite-to-one factor map.

- Adler and Weiss 1967 for the case of dimension $d = 2$.
- Sinai 1968 any finite dimension d.
- Bowen 1970, for basic sets of Axiom A diffeomorphisms.

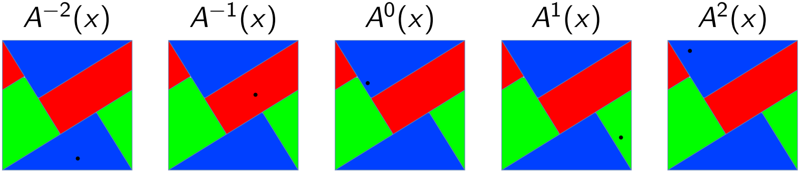




Markov Property



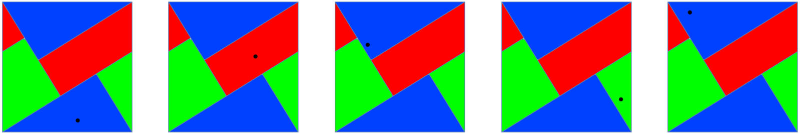
Tracking orbits



e_1.

Tracking orbits

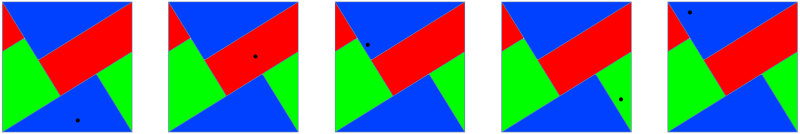
$A^{-2}(x)$ $A^{-1}(x)$ $A^{0}(x)$ $A^{1}(x)$ $A^{2}(x)$



e_3 $e_1.$ e_2

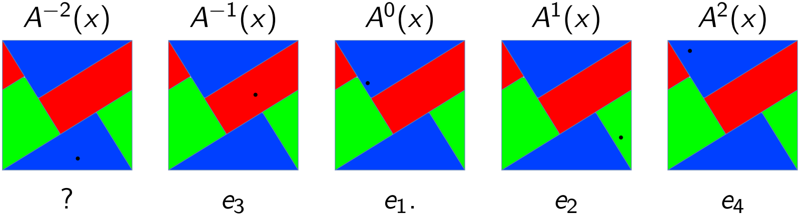
Tracking orbits

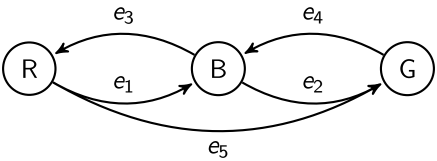
$A^{-2}(x)$ $A^{-1}(x)$ $A^{0}(x)$ $A^{1}(x)$ $A^{2}(x)$



e_3 $e_1.$ e_2

Tracking orbits





2nd example-shifts of finite type

Let G be a finite directed graph which consists of a vertex set G^0, an edge set G^1, and two maps $r, s : G^1 \to G^0$. The source vertex of edge e is given by $s(e)$ and the range vertex is given by $r(e)$.

Definition

We define

$$\Sigma_G = \{(x_n)_{n\in\mathbb{Z}} \mid x_n \in G^1,\ r(x_n) = s(x_{n+1}) \text{ for all } n \text{ in } \mathbb{Z}\}$$

With the left shift map $\sigma : \Sigma_G \to \Sigma_G$,

$$\sigma(x)_n = x_{n+1}.$$

Definition

A factor map π, has a splitting, if it is a composition of a u and s-bijective map.

$$\begin{tikzcd}
 & (\Sigma_G, \sigma) \arrow[dl, dashed, "\mu_s"'] \arrow[dd, "\pi"] \\
(\Omega, \omega) \arrow[dr, dashed, "\rho_u"'] & \\
 & (\mathbb{T}^d, A)
\end{tikzcd}$$

Definition

We say that $\pi : (X, f) \to (Y, g)$ is s-bijective if, for any x in X, its restriction to $X^s(x)$ is a bijection to $Y^s(\pi(x))$.

Theorem

Let $\pi : (X, f) \to (Y, g)$ be an s-bijective map. Then for every x in X, the map $\pi : X^s(x, \epsilon) \to Y^s(\pi(x), \epsilon')$ is a local homeomorphism.

A u-bijective map is defined and characterized analogously.

Suppose we also have μ_s, an s-bijective map and ρ_u, a u-bijective map such that,

$$\begin{tikzcd}
 & (\Sigma_G, \sigma) \arrow[dl, "\mu_s"'] \arrow[dd, "\pi"] \\
(\Omega, \omega) \arrow[dr, "\rho_u"'] & \\
 & (\mathbb{T}^d, A)
\end{tikzcd}$$

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$$\begin{tikzcd}
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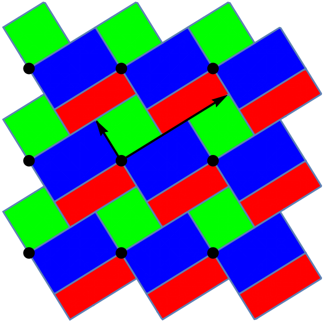
(\Omega, \omega) \arrow[dr, "\rho_u"'] & \\
 & (\mathbb{T}^d, A)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{Cantor} \times \text{Cantor} \arrow[dl, "\mu_s"'] \arrow[dd, "\pi"] \\
??? \arrow[dr, "\rho_u"'] & \\
 & \mathbb{R}^m \times \mathbb{R}^n \cong E^s \times E^u
\end{tikzcd}$$

What must the intermediary space look like locally?
What is a candidate space?

(Ω, ω) is a Tiling space!

$$
\begin{tikzcd}
 & (\Sigma_{G_{\mathrm{fib1}}}, \sigma) \arrow[dl, "\mu_s"'] \arrow[dd, "\pi"] \arrow[dr, "\mu_u"] & \\
(\Omega_1, \omega_1) \arrow[dr, "\rho_u"] & & (\Omega', \omega') \arrow[dl, "\rho_s"] \\
 & (\mathbb{T}^2, A) &
\end{tikzcd}
$$



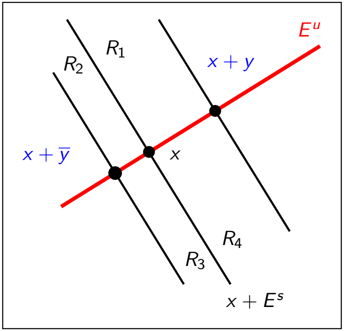
Definition

A factor map π, has a splitting, if it is a composition of a u and s-bijective map.

$$
\begin{tikzcd}
 & (\Sigma_G, \sigma) \arrow[ld, dashed, "\mu_s"'] \arrow[dd, "\pi"] \\
(\Omega, \omega) \arrow[rd, dashed, "\rho_u"'] & \\
 & (\mathbb{T}^d, A)
\end{tikzcd}
$$

Does the splitting always exist? Is there a necessary and sufficient condition for a given factor map, π, to have a splitting?

A splitting for the map π exists if the boundary around a periodic point looks something like this...

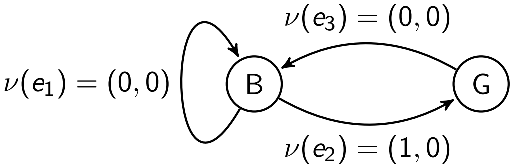


The stable boundaries around a periodic point (for the map A) should look the same in the E^u direction.

Constructing Markov partitions

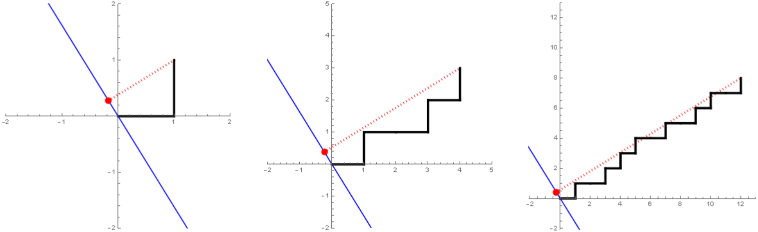
Let G_{fib} be the following finite directed graph with labelling map,

$$\nu : G^1_{\text{fib}} \to \mathbb{Z}^2.$$



Suppose we take $x = \dots e_2 e_3 e_2 e_3 e_2 e_3 . e_2 e_3 e_2 e_3 e_2 e_3 \dots$ from $\Sigma_{G_{\text{fib}}}$.

$$\pi^s(x) = \sum_{n \leq 0} A^{-n}\nu(x_n)^s = \lim_{k\to\infty}\left(\sum_{n=0}^{-k} A^{-n}\nu(x_n)\right)^s$$

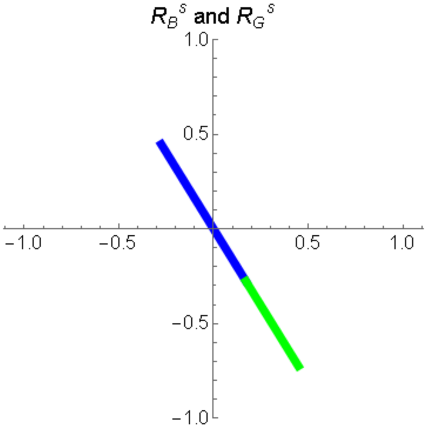


k=1 k=3 k=5

$$A(1,0) + A^3(1,0) + A^5(1,0)\ldots$$

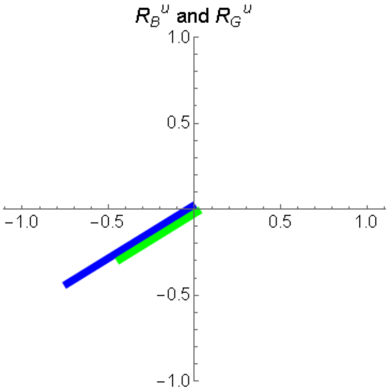
$$R_B^s = \pi^s\{x \in \Sigma_{G_{\text{fib}}} \mid r(x_0) = B\}$$
$$= \text{blue set}$$

$$R_G^s = \pi^s\{x \in \Sigma_{G_{\text{fib}}} \mid r(x_0) = G\}$$
$$= \text{green set}$$

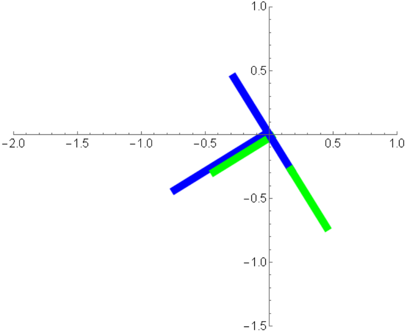


$$R_B^u = \pi^u\{x \in \Sigma_{G_{\text{fib}}} \mid r(x_0) = B\}$$
$$= \text{blue set}$$

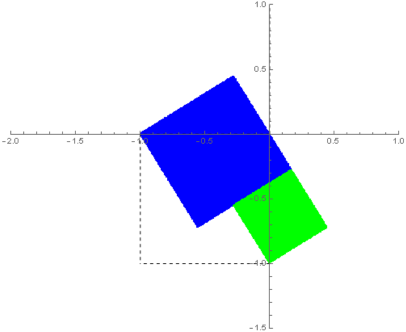
$$R_G^u = \pi^u\{x \in \Sigma_{G_{\text{fib}}} \mid r(x_0) = G\}$$
$$= \text{green set}$$



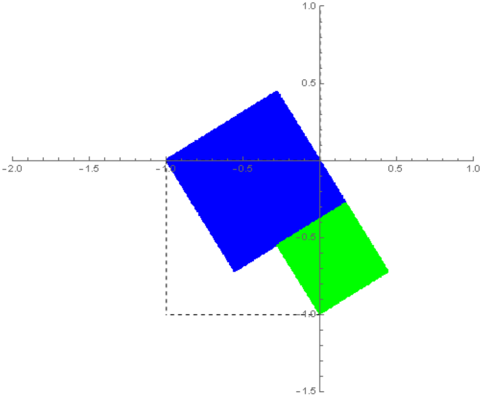
$R^s_B, R^s_G, R^u_B, R^u_G$



$R^s_B + R^u_B,\ R^s_G + R^u_G$.

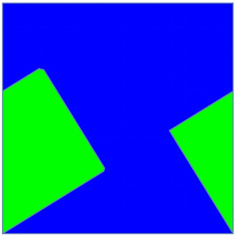


$R_B^s + R_G^u,\ R_G^s + R_G^u$



$\mathcal{R}$

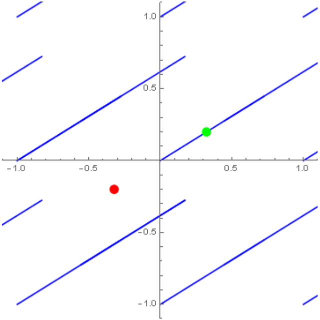
$q : \mathbb{R}^2 \to \mathbb{T}^2$ mod $\mathbb{Z}^2$ map.



$q(\mathcal{R})$

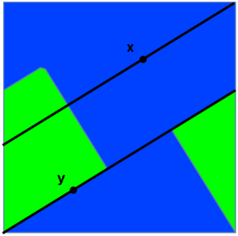
Our G_{fib} example does not satisfy the condition

$\partial^s \mathcal{R}$



Idea for proof

Suppose the condition fails.



Choose x and y unstably equivalent and stably equivalent to a periodic point, where x has one preimage under π while y has two preimages under π. Contradicts properties of u and s-bijective maps. No splitting for the map $\pi : \Sigma_{G_{\text{fib}}} \to \mathbb{T}^2$ exists.

Does there exist another SFT for which the factor map splits?

Theorem (Putnam,2005)

Let (Y, g) be an irreducible Smale space. Then there exists a shift of finite type (Σ, σ), another irreducible Smale space (Ω, ω), and

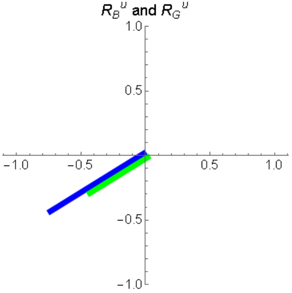
$$\mu : (\Sigma, \sigma) \to (\Omega, \omega)$$
$$\rho : (\Omega, \omega) \to (Y, g)$$

factor maps, such that μ is s-bijective and ρ is u-bijective.

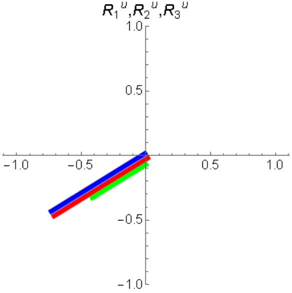
How do we find it, explicitely?

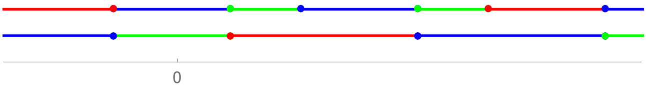
The collared tiling system $(\Omega_1, \mathcal{P}_1, \omega_1)$ forces its border.

Original prototiles



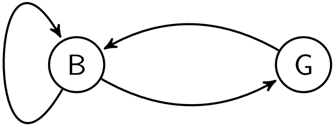
Collared prototiles



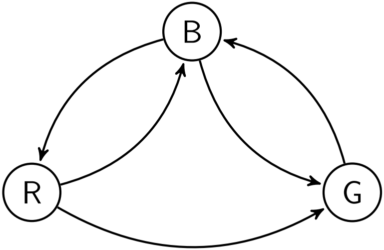


Non-conjugate shifts of finite type

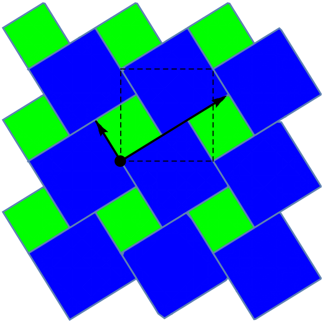
$(\Sigma_{G_{\text{fib}}}, \sigma)$ (old)



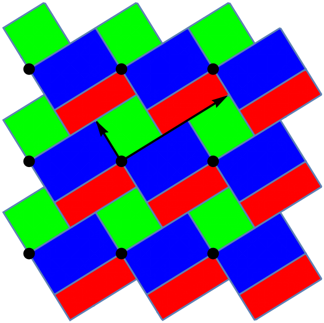
$(\Sigma_{G_{\text{fib1}}}, \sigma)$ (new)



New Markov partition.



$\Longrightarrow$



From Anderson and Putnam 1998 and Wieler 2005.

$$
\begin{tikzcd}
 & (\Sigma_{G_{\text{fib1}}}, \sigma) \arrow[dl, "\mu_s"'] \arrow[dd, "\pi"] \arrow[dr, "\mu_u"] & \\
(\Omega_1, \omega_1) \arrow[dr, "\rho_u"] & & (\Omega', \omega') \arrow[dl, "\rho_s"] \\
 & (\mathbb{T}^2, A) &
\end{tikzcd}
$$

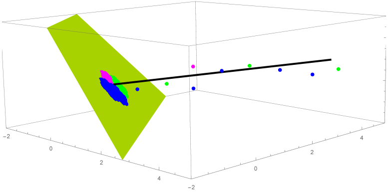
A three dimensional example

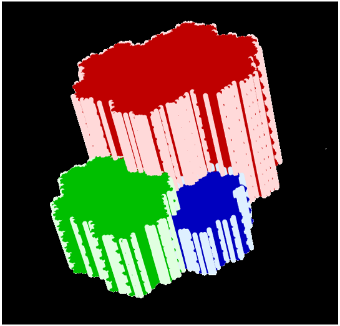
Let $\hat{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. The induced map B defines an HTA of $\mathbb{T}^3$.

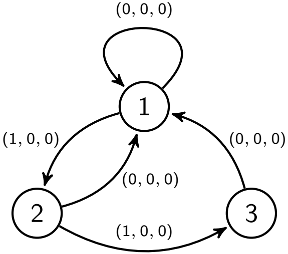
Eigenvalues: $\beta > 1$, $\alpha, \overline{\alpha}$, where $\beta^3 - \beta^2 - \beta - 1 = 0$.

Expanding line and contracting plane.

The Markov partition is given by the following (viewed in $\mathbb{R}^3$).







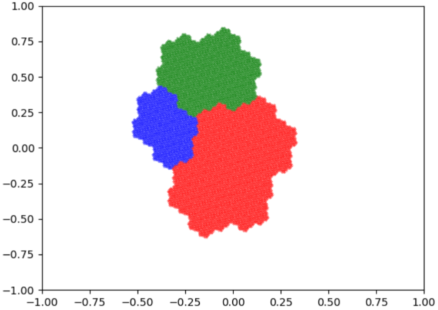
Unstable (Tribonacci)

$$AR_1^u = R_1^u \cup R_2^u - (1,0,0)^s$$
$$AR_2^u = R_1^u \cup R_3^u - (1,0,0)^s$$
$$AR_3^u = R_1^u$$

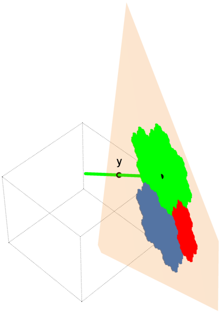
Stable (Rauzy)

$$R_1^s = AR_1^s \cup AR_2^s \cup AR_3^s$$
$$R_2^s = AR_1^s + (1,0,0)^s$$
$$R_3^s = AR_2^s + (1,0,0)^s$$

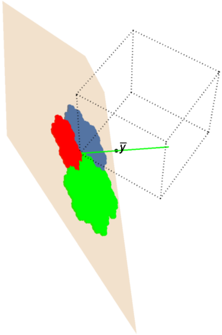
$R_1^s = AR_1^s \cup AR_2^s \cup AR_3^s$



Interior point



Boundary point



No splitting for the factor map exists.

$$\begin{tikzcd}
& (\Sigma_{G_{\text{new}}}, \sigma) \arrow[dl, "\mu_s"] \arrow[dr, "\mu_u"] \arrow[dd, "\pi_{\text{new}}"] & \\
(\Omega_1, \omega_1) \arrow[dr, "\rho_u"] & & (\Omega', \omega') \arrow[dl, "\rho_s"] \\
& (\mathbb{T}^2, A) &
\end{tikzcd}$$

(Ω_1, ω_1)	collared tribonacci substitution
(Ω', ω')	collared Rauzy substitution (tiling example 1)

We know:

- Existence of splitting for $\pi \implies$ condition on boundaries of MP.
- Forcing the border of $(\Omega, \omega) \implies \exists$ a map π that splits.
- We have been working on an example of a factor map that splits, but the correspdonding tiling system (Ω, ω) does not force its border.

Questions:

- Does the condition being satisfied imply the existence of a splitting?
- If we randomly label a graph of the SFT what sort of sets in $\mathbb{R}^d$ are possible? Under which conditions?
- What does all of this have to do with Ian's homology theory for Smale spaces?

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Thank you for your attention!